

CRITICAL POINTS FOR TWO-VIEW TRIANGULATION

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ABSTRACT. Two-view triangulation is a problem of minimizing a quadratic polynomial under an equality constraint. We derive a polynomial that encodes the local minimizers of this problem using the theory of Lagrange multipliers. This offers a simpler derivation of the critical points that are given in Hartley-Sturm [6].

1. INTRODUCTION

Two-view triangulation is the problem of estimating a point $X \in \mathbb{R}^3$ from two noisy image projections; see [5, Chapter 12] for its significance in structure from motion in computer vision. Assuming a Gaussian error distribution, one way to solve the problem is to compute the maximum likelihood estimates (MLE) for the true image point correspondences. After that the point $X \in \mathbb{R}^3$ can be recovered via linear algebra [5]. In this paper we study the above problem of finding the MLEs. According to the discussion in [1] or [5, Chapter 12], the problem is formulated as follows.

Consider a rank two matrix $F \in \mathbb{R}^{3 \times 3}$ which is called a *fundamental matrix* in multi-view geometry. This matrix F encodes a pair of projective cameras [5, Chapter 9]. Given two points $u_1, u_2 \in \mathbb{R}^2$ which denote the noisy image projections, we solve the problem

$$(1.1) \quad \begin{aligned} & \min_{x_1, x_2 \in \mathbb{R}^2} \|x_1 - u_1\|_2^2 + \|x_2 - u_2\|_2^2 \\ & \text{subject to } \hat{x}_2^\top F \hat{x}_1 = 0 \end{aligned}$$

where $\hat{x}_k := (x_k^\top \ 1)^\top \in \mathbb{R}^3$ for $k = 1, 2$. The equation $\hat{x}_2^\top F \hat{x}_1 = 0$ is called the *epipolar constraint*, which indicates that x_1 and x_2 are the true image projections under the projective cameras associated with F . The minimizers of (1.1) are the MLEs for the true image correspondences, assuming the error is Gaussian.

In [5, Chapter 12] (or [6]) there is a technique for finding the global minimizers of (1.1) using a non-iterative approach. They use multi-view geometry to reformulate the problem (1.1) as minimizing a fraction in a single real variable say t . Using the Fermat rule in elementary calculus, it turns out that the minimizers can be computed via finding the real roots of a polynomial in t of degree 6.

In this note, we view the problem (1.1) as minimizing a multivariate quadratic polynomial over one single equality constraint, and then employ the classical method of Lagrange multipliers to locate the potential local minimizers. These candidates are called *critical points*. For general rank two matrices F and general points u_1, u_2 , there are six critical points. They can be computed via finding the roots of a polynomial of degree 6 in the Lagrange multiplier. Assuming that a global minimizer exists, the minimizer of (1.1) can be obtained from the critical points.

2. SIX CRITICAL POINTS FOR TWO-VIEW TRIANGULATION

2.1. Reformulation of the problem (1.1). Given a fundamental matrix $F \in \mathbb{R}^{3 \times 3}$ and $u_1 = \begin{pmatrix} u_{11} & u_{12} \end{pmatrix}^\top$, $u_2 = \begin{pmatrix} u_{21} & u_{22} \end{pmatrix}^\top \in \mathbb{R}^2$, consider the invertible matrices $W_1 := \begin{pmatrix} 1 & 0 & -u_{11} \\ 0 & 1 & -u_{12} \\ 0 & 0 & 1 \end{pmatrix}$ and $W_2 := \begin{pmatrix} 1 & 0 & -u_{21} \\ 0 & 1 & -u_{22} \\ 0 & 0 & 1 \end{pmatrix}$. Note that $\|x_k - u_k\|^2 = \|\hat{x}_k - \hat{u}_k\|^2$, and that problem (1.1) is equivalent to the problem

$$\begin{aligned} \min_{x_1, x_2 \in \mathbb{R}^2} \quad & \|W_1 \hat{x}_1\|_2^2 + \|W_2 \hat{x}_2\|_2^2 \\ \text{subject to} \quad & \hat{x}_2^\top F \hat{x}_1 = 0 \end{aligned}$$

For all $k = 1, 2$, the last coordinate of $W_k \hat{x}_i$ equals one. As a result, we let $y_k \in \mathbb{R}^2$ be such that $\hat{y}_k = W_k \hat{x}_k$. Then (1.1) is further equivalent to the problem

$$\begin{aligned} \min_{y_1, y_2 \in \mathbb{R}^2} \quad & \frac{1}{2} (\|\hat{y}_1\|_2^2 + \|\hat{y}_2\|_2^2) \\ \text{subject to} \quad & \hat{y}_2^\top F' \hat{y}_1 = 0 \end{aligned} \quad (2.1)$$

where $F' := W_2^{-\top} F W_1^{-1} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ is another fundamental matrix.

2.2. Derivation of a six degree polynomial. Let $G(y_1, y_2) := \frac{1}{2} (\|\hat{y}_1\|_2^2 + \|\hat{y}_2\|_2^2)$ and $H(y_1, y_2) := \hat{y}_2^\top F' \hat{y}_1$. The Karush-Kuhn-Tucker (KKT) equation for (2.1) is $\nabla G + \lambda \nabla H = 0$ for some $\lambda \in \mathbb{C}$ called the Lagrange multiplier; see any nonlinear programming text e.g. [2]. Unwinding this equation we obtain a linear system in four variables, namely,

$$(2.2) \quad \begin{pmatrix} 1 & 0 & \lambda a & \lambda b \\ 0 & 1 & \lambda d & \lambda e \\ \lambda a & \lambda d & 1 & 0 \\ \lambda b & \lambda e & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{21} \\ y_{22} \\ y_{11} \\ y_{12} \end{pmatrix} = -\lambda \begin{pmatrix} c \\ f \\ g \\ h \end{pmatrix}$$

where $y_k = \begin{pmatrix} y_{k1} & y_{k2} \end{pmatrix}^\top$ for $k = 1, 2$, and λ is the Lagrange multiplier. To acquire the critical points we derive a polynomial equation in λ . It comes from first expressing y_k , $k = 1, 2$, in terms of u_1, u_2, F and then substituting these expressions into the epipolar constraint $\hat{y}_2^\top F' \hat{y}_1 = 0$. Let A_λ be the 4×4 coefficient matrix of the above system. One has

$$\det(A_\lambda) = (bd - ae)^2 \lambda^4 - (a^2 + b^2 + d^2 + e^2) \lambda^2 + 1.$$

Define $p_{kl} := \det(A_\lambda) y_{kl}$ for $k, l = 1, 2$. By Cramer's rule one has

$$\begin{aligned} p_{21} &= \lambda[(bd - ae)(eg - dh)\lambda^3 + (d^2c + e^2c - adf - bef)\lambda^2 + (ag + bh)\lambda - c] \\ p_{22} &= \lambda[(bd - ae)(ah - bg)\lambda^3 + (a^2f + b^2f - acd - bce)\lambda^2 + (dg + eh)\lambda - f] \\ p_{11} &= \lambda[(bd - ae)(ce - bf)\lambda^3 + (b^2g + e^2g - abh - deh)\lambda^2 + (ac + df)\lambda - g] \\ p_{12} &= \lambda[(bd - ae)(af - cd)\lambda^3 + (a^2h + d^2h - abg - deg)\lambda^2 + (bc + ef)\lambda - h]. \end{aligned}$$

Consider the polynomial

$$T := -\det(A_\lambda)^2 \hat{y}_2^\top F' \hat{y}_1 = -p_2^\top F' p_1$$

where $p_k := \begin{pmatrix} p_{k1} & p_{k2} & \det(A_\lambda) \end{pmatrix}^\top$ for $k = 1, 2$. Since $\det(A_\lambda)$ is a quartic in λ , and p_{kl} is also a quartic in λ for $k, l = 1, 2$, we know T is a polynomial in λ of

degree at most 8. By a careful and slightly tedious computation without using any machines, or by using the following Macaulay2 [4] code:

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R = QQ[a,b,c,d,e,f,g,h,i,L];
A = matrix{{1,0,L*a,L*b},{0,1,L*d,L*e},{L*a,L*d,1,0},{L*b,L*e,0,1}};
detA = det A;
p21 = det matrix{{-L*c,0,L*a,L*b},{-L*f,1,L*d,L*e},{-L*g,L*d,1,0},{-L*h,L*e,0,1}};
p22 = det matrix{{1,-L*c,L*a,L*b},{0,-L*f,L*d,L*e},{L*a,-L*g,1,0},{L*b,-L*h,0,1}};
p11 = det matrix{{1,0,-L*c,L*b},{0,1,-L*f,L*e},{L*a,L*d,-L*g,0},{L*b,L*e,-L*h,1}};
p12 = det matrix{{1,0,L*a,-L*c},{0,1,L*d,-L*f},{L*a,L*d,1,-L*g},{L*b,L*e,0,-L*h}};
T = -(a*p11*p21+b*p12*p21+c*p21*detA+d*p11*p22+
      e*p12*p22+f*p22*detA+g*p11*detA+h*p12*detA+i*detA*detA);
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we know the coefficient of λ^7 is zero. The coefficient of λ^8 is

$$\begin{aligned} & -(bd - ae)^2(eg - dh)(ace - abf + baf - bcd + cbd - cae) + \\ & -(bd - ae)^2(ah - bg)(dce - dbf + eaf - ecd + fbd - fae) + \\ & -(bd - ae)^3(gce - gbh + haf - hcd + ibd - iae) = (bd - ae)^3 \det(F) = 0 \end{aligned}$$

since F has rank two. This implies T is a polynomial in λ of degree at most six. Here we record the explicit expression of T :

$$\begin{aligned} T = & (bd - ae)^2(acg + dfh + bch + efg - a^2i - b^2i - d^2i - e^2i)\lambda^6 + \\ & a^2c^2d^2\lambda^5 + c^2d^4\lambda^5 + 2abc^2de\lambda^5 + b^2c^2e^2\lambda^5 + 2c^2d^2e^2\lambda^5 + c^2e^4\lambda^5 - \\ & 2a^3cdf\lambda^5 - 2ab^2cdf\lambda^5 - 2acd^3f\lambda^5 - 2a^2bce f\lambda^5 - 2b^3cef\lambda^5 - 2bcd^2ef\lambda^5 - \\ & 2acde^2f\lambda^5 - 2bce^3f\lambda^5 + a^4f^2\lambda^5 + 2a^2b^2f^2\lambda^5 + b^4f^2\lambda^5 + a^2d^2f^2\lambda^5 + \\ & 2abdef^2\lambda^5 + b^2e^2f^2\lambda^5 + a^2b^2g^2\lambda^5 + b^4g^2\lambda^5 + 2abdeg^2\lambda^5 + 2b^2e^2g^2\lambda^5 + \\ & d^2e^2g^2\lambda^5 + e^4g^2\lambda^5 - 2a^3bgh\lambda^5 - 2ab^3gh\lambda^5 - 2abd^2gh\lambda^5 - 2a^2degh\lambda^5 - \\ & 2b^2degh\lambda^5 - 2d^3egh\lambda^5 - 2abe^2gh\lambda^5 - 2de^3gh\lambda^5 + a^4h^2\lambda^5 + a^2b^2h^2\lambda^5 + \\ & 2a^2d^2h^2\lambda^5 + d^4h^2\lambda^5 + 2abdeh^2\lambda^5 + d^2e^2h^2\lambda^5 + a^3cg\lambda^4 + ab^2cg\lambda^4 + \\ & acd^2g\lambda^4 - 5bcdeg\lambda^4 + 6ace^2g\lambda^4 + a^2dfg\lambda^4 + 6b^2dfg\lambda^4 + d^3fg\lambda^4 - \\ & 5abefg\lambda^4 + de^2fg\lambda^4 + a^2bch\lambda^4 + b^3ch\lambda^4 + 6bcd^2h\lambda^4 - 5acdeh\lambda^4 + \\ & bce^2h\lambda^4 - 5abdfh\lambda^4 + 6a^2efh\lambda^4 + b^2efh\lambda^4 + d^2efh\lambda^4 + e^3fh\lambda^4 - a^4i\lambda^4 - \\ & 2a^2b^2i\lambda^4 - b^4i\lambda^4 - 2a^2d^2i\lambda^4 - 4b^2d^2i\lambda^4 - d^4i\lambda^4 + 4abdei\lambda^4 - 4a^2e^2i\lambda^4 - \\ & 2b^2e^2i\lambda^4 - 2d^2e^2i\lambda^4 - e^4i\lambda^4 - 2c^2d^2\lambda^3 - 2c^2e^2\lambda^3 + 4acdf\lambda^3 + 4bcef\lambda^3 - \\ & 2a^2f^2\lambda^3 - 2b^2f^2\lambda^3 - 2b^2g^2\lambda^3 - 2e^2g^2\lambda^3 + 4abgh\lambda^3 + 4degh\lambda^3 - 2a^2h^2\lambda^3 - \\ & 2d^2h^2\lambda^3 - 3acg\lambda^2 - 3dfg\lambda^2 - 3bch\lambda^2 - 3efh\lambda^2 + 2a^2i\lambda^2 + 2b^2i\lambda^2 + \\ & 2d^2i\lambda^2 + 2e^2i\lambda^2 + c^2\lambda + f^2\lambda + g^2\lambda + h^2\lambda - i. \end{aligned}$$

2.3. The six critical points. By solving $T = 0$ for λ , we get six (complex) solutions (counting multiplicities) for λ , say $\lambda_1, \dots, \lambda_6$. Plugging in these six values of λ into the linear system (2.2), solving the linear system for y_1 and y_2 , and computing x_1 and x_2 , one obtains the critical points for two-view triangulation. If $\det(A_{\lambda_k}) \neq 0$ for every $k = 1, \dots, 6$ then there are precisely six critical points counting multiplicities.

x_{21}	x_{22}	x_{11}	x_{12}
0.0596	-0.0321	-0.312	-0.891
-0.0843	-2.06	-0.438	-0.0259
$-2.42 + 0.0137i$	$-1.02 - 1.56i$	$-1.57 + 0.714i$	$-1.246 - 1.51i$
$-2.42 - 0.0137i$	$-1.02 + 1.56i$	$-1.57 - 0.714i$	$-1.246 + 1.51i$
$-1.69 + 0.0226i$	$-0.935 + 0.414i$	$0.748 + 0.169i$	$-0.279 - 0.574i$
$-1.69 - 0.0226i$	$-0.935 - 0.414i$	$0.748 - 0.169i$	$-0.279 + 0.574i$

TABLE 1. Six critical points for (1.1) when $u_1 = (0 \ 0)^\top$, $u_2 = u_1$ and $F = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 3 & 3 \end{pmatrix}$.

Now we claim that for general fundamental matrices F and points $u_1, u_2 \in \mathbb{R}^2$, there are six distinct critical points for two-view triangulation. The claim is false if and only if the discriminant of T or the resultant of T and $\det(A_\lambda)$ are zero polynomials. Instead of computing the desired discriminant and resultant which depend on u_1, u_2 and F , one can find an example of (u_1, u_2, F) such that the discriminant of T and the resultant of T and $\det(A_\lambda)$ take a nonzero value, that is, $\det(A_\lambda) \neq 0$ for every solution λ of T , and the six critical points obtained are distinct. If we consider the data $u_1 = (0 \ 0)^\top$, $u_2 = u_1$ and $F = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 3 & 3 \end{pmatrix}$, then the polynomial T becomes $-2\lambda^6 + 6\lambda^5 + 3\lambda^4 - 12\lambda^3 - 3\lambda^2 + 12\lambda - 3$, and there are six distinct complex critical points for the problem (1.1); see Table 1.

We summarize the discussion in the following theorem.

Theorem 2.1. *For general points $u_1, u_2 \in \mathbb{R}^2$ and fundamental matrices F , there are six complex critical points for the problem (1.1).*

3. DISCUSSION

One can make sense of the critical points for n -view triangulation where n is greater than two. The authors in [8] (cf. [7]) computed the number of critical points for 2 to 7 view triangulation are 6, 47, 148, 336, 638, 1081. Draisma et al. [3] call this list of numbers the *Euclidean distance degrees* of the multi-view variety associated to 2 to 7 cameras. They conjecture that the general term of this sequence is

$$C(n) := \frac{9}{2}n^3 - \frac{21}{2}n^2 + 8n - 4.$$

One can apply the Bézout's theorem to conclude that $C(n)$ has order n^3 , and our paper verified $C(2) = 6$. However a proof of the above general formula is still unknown.

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